On Generalisations of the AVD Conjecture to Digraphs

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Abstract

Given an undirected graph, in the AVD (edge-colouring) Conjecture, the goal is to find a proper edge-colouring with the least number of colours such that every two adjacent vertices are incident to different sets of colours. More precisely, the conjecture says that, a few exceptions apart, every graph G should admit such an edge-colouring with at most $\Delta(G)+2$ colours. Several aspects of interest behind this problem have been investigated over the recent years, including verifications of the conjecture for particular graph classes, general approximations of the conjecture, and multiple generalisations.

In this paper, following a recent work of Sopena and Woźniak, generalisations of the AVD Conjecture to digraphs are investigated. More precisely, four of the several possible ways of generalising the conjecture are focused upon. We completely settle one of our four variants, while, for the three remaining ones, we provide partial results.

Keywords: AVD Conjecture; proper edge-colourings; digraphs.

1. Introduction

One of the most central notions of graph theory is that of proper edge-colourings. Given an undirected graph G, a *proper k-edge-colouring* ϕ of G is an assignment $\phi : E(G) \rightarrow \{1, \dots, k\}$ of colours to the edges such that no two adjacent edges (*i.e.*, incident to a same vertex) get assigned the same colour. We are usually interested in determining the *chromatic index* $\chi'(G)$ of G, which refers to the smallest k such that G admits a proper k-edge-colouring. Perhaps the most important result regarding the chromatic index of graphs is Vizing's Theorem [16], which states that, for every graph G, we have $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ (where $\Delta(G)$ denotes the maximum degree of a vertex in G). Even though that result means the chromatic index of any graph is one of only two possible values, it is important to recall that determining the chromatic index of a graph is an NP-complete problem in general [10].

In several contexts, it might be convenient to have edge-colourings of graphs that are not only proper, but also have additional properties. When considering such a stronger form of proper edge-colourings, an interesting question is about the least number of additional colours needed to construct one for any given graph.

In this work, we are mostly interested in proper edge-colourings that allow to distinguish adjacent vertices according to their respective sets of incident colours. More precisely,

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given a proper edge-colouring ϕ of a graph G, for every vertex u, we can compute $\mathrm{S}(u)$, which is the set of colours assigned by ϕ to the edges incident to u. Note that, by the properness of ϕ, we always have $|\mathrm{S}(u)| = d(u)$. Now, we say that ϕ is *distinguishing* if, for every edge uv, we have $\mathrm{S}(u) \neq \mathrm{S}(v)$. We denote by $\mathrm{ndi}(G)$ (where "ndi" stands for "neighbour-distinguishing index") the least k such that G admits a distinguishing proper k-edge-colouring (if any). Actually, it is easy to see that $\mathrm{ndi}(G)$ is defined if and only if G has no connected component isomorphic to K_2 (just consider a proper edge-colouring assigning a distinct colour to every edge). Thus, regarding those notions, we are interested in *nice graphs*, which are the graphs with no connected component isomorphic to K_2.

We employ the simple terminology and notations above for the sake of the current work's legibility. It is worth mentioning, however, that this terminology and these notations vary from one work to another in the literature. In particular, our notion of distinguishing edge-colouring is sometimes called *adjacent vertex-distinguishing edge-colouring*, *neighbour-distinguishing edge-colouring*, *adjacent strong edge-colouring* or 1-*strong edge-colouring*. Our parameter ndi is sometimes written χ'_{avd} and called the *adjacent vertex-distinguishing chromatic index*. Even more different terms and notations are used in some works.

Clearly, we have $\chi'(G) \leq \mathrm{ndi}(G)$ for every nice graph G. Regarding the concerns above, a natural question to ask is how large can $\mathrm{ndi}(G)$ be in general. It can be noted that $\mathrm{ndi}(C_5) = \Delta(C_5) + 3 = 5$, where C_5 denotes the cycle of length 5. However, it is believed that C_5 should be the only nice graph G for which $\mathrm{ndi}(G)$ is that large. This leads to the following conjecture raised by Zhang, Liu, and Wang in 2002 [19].

AVD Conjecture. *For every nice connected graph $G \neq C_5$, we have* $\mathrm{ndi}(G) \leq \Delta(G) + 2$.

Comparing the AVD Conjecture to Vizing's Theorem, the conjecture indicates that, in general, at most two additional colours might be necessary to turn a normal proper $\chi'(G)$-edge-colouring of a graph G into a distinguishing one.

Several interesting results towards the AVD Conjecture have been obtained since its introduction. In particular, the conjecture was verified for bipartite graphs and subcubic graphs [2]. It was also proven in [1], that every nice graph G verifies $\mathrm{ndi}(G) \leq 3\Delta(G)$. We refer the interested reader to [14] for more details. As a particular point, it is important to mention again that the investigations around the AVD Conjecture are part of a wider family of problems, where one aims at designing proper vertex-colourings through (not necessarily proper) edge-colourings. In particular, [7, 8] can be considered as the starting points of many interesting investigations later on.

In graph theory, a common line of research is, given a particular problem defined on undirected graphs, to wonder about its counterparts on digraphs. Regarding the AVD Conjecture, this is a promising prospect due to the numerous generalisations that can be considered. Indeed, by a proper arc-colouring of a digraph D, note that every vertex u can now be associated two sets $\mathrm{S}^+(u)$ and $\mathrm{S}^-(u)$ of incident colours, where $\mathrm{S}^+(u)$ is the set of colours assigned to the arcs out-going from u, and $\mathrm{S}^-(u)$ is the set of colours assigned to the arcs in-coming to u. Recall that, in the directed context, a *proper arc-colouring* ϕ of D verifies that two arcs out-going from a same vertex are assigned distinct colours, and similarly for two arcs in-coming to a same vertex. Under that definition, note that an arc out-going from u and an arc in-coming to u can be assigned the same colour. Note also that, because ϕ is proper, we have $|\mathrm{S}^+(u)| = d^+(u)$ and $|\mathrm{S}^-(u)| = d^-(u)$, which is reminiscent of the similar property of proper edge-colourings of undirected graphs.

For a digraph D, its *chromatic index* $\chi'(D)$ is the least k such that D admits proper k-arc-colourings. By the definition of proper arc-colourings, note that we always have $\chi'(D) \geq \Delta^*(D) = \max\{\Delta^+(D), \Delta^-(D)\}$, where $\Delta^+(D)$ and $\Delta^-(D)$ denote the maximum out-degree and maximum in-degree, respectively, over all vertices of D. In contrast with the undirected context of Vizing's Theorem, it is known that the chromatic index of a digraph D is always precisely the natural lower bound $\Delta^*(D)$ (see, *e.g.* [18] or the proof of Observation 3.2 later).

Since every vertex gets assigned two sets of colours in any proper arc-colouring ϕ of a digraph, there are plenty of ways of considering that two adjacent vertices are distinguished by ϕ, and thus, many possibilities for defining a directed counterpart to the AVD Conjecture. To the best of our knowledge, this line of research was not considered until quite recently, by Sopena and Woźniak in [15]. In their variant of the problem, they consider that two adjacent vertices u and v are distinguished by a proper arc-colouring when either $\mathrm{S}^+(u) \neq \mathrm{S}^+(v)$ or $\mathrm{S}^-(u) \neq \mathrm{S}^-(v)$. They conjectured that every digraph D admits such a distinguishing proper arc-colouring using at most $\Delta^*(D)+1$ colours, and they proved that $2\Delta^*(D)$ colours are enough to construct one.

Let us mention that other related problems generalised to digraphs were considered before the work of Sopena and Woźniak. In particular, the series [3, 4, 5, 6, 11] of works, dedicated to directed variants of the so-called 1-2-3 Conjecture, are very close to the investigations conducted in the current paper. It turns out, actually, that some of the phenomena observed through our results and the proof techniques we develop, are reminiscent of some from these works.

In the line of the investigations initiated by Sopena and Woźniak, we consider several directed variants of the AVD Conjecture. Specifically, we consider four variants, in which, for every two adjacent vertices, we ask that a single of the two set parameters differs. Precisely, our general terminology is as follows. Let D be a digraph, and ϕ a proper arc-colouring of D. To each of the set parameters S^+ and S^-, we associate a sign, namely $+$ and $-$, respectively. Now, for any two signs $\alpha, \beta \in \{+,-\}$, we say that ϕ is (α,β)-*distinguishing* if, for every arc $\vec{uv}$ of D, the set parameter of u associated to α is different from the set parameter of v associated to β. We denote by $\mathrm{ndi}_{\alpha,\beta}(D)$ the least k such that D admits an (α,β)-distinguishing proper k-arc-colouring (if any).

Note that this general terminology encapsulates a series of four colouring problems, each of which has behaviours that are more or less reminiscent of the original AVD Conjecture. In terms of colouring behaviours, note that the $(+,-)$ version is the closest to the original problem, as, by a proper arc-colouring of a digraph, assigning a colour to an arc $\vec{uv}$ affects $\mathrm{S}^+(u)$ and $\mathrm{S}^-(v)$ which are precisely the two set parameters that are required to be different. From this perspective, the $(+,+)$ and $(-,-)$ versions are a bit farther from the original conjecture, while the $(-,+)$ version is the most distant. Still, for all four of the versions, recall that the number of colours needed in a distinguishing proper arc-colouring is strongly dependent on the chromatic index.

Observation 1.1. *Let $\alpha, \beta \in \{+,-\}$. For every digraph D such that $\mathrm{ndi}_{\alpha,\beta}(D)$ is defined, we have $\chi'(D) \leq \mathrm{ndi}_{\alpha,\beta}(D)$.*

Although our four variants of the AVD Conjecture have their own peculiar behaviours, we feel that, in general, for any of the variants, $\Delta^*(D)+2$ colours should always be enough to construct a desired distinguishing proper arc-colouring. Our goal in this paper is to provide evidence towards this intuition. Section 2 is dedicated to investigating the $(+,+)$ and $(-,-)$ versions of the AVD Conjecture, while Section 3 is dedicated to the $(+,-)$ version, and Section 4 is dedicated to the $(-,+)$ variant. We provide a tight result on the $(+,-)$ variant, while we provide partial results for the other variants.

2. The $(+,+)$ and $(-,-)$ versions

Note that a $(+,+)$-distinguishing proper arc-colouring of a given digraph D directly yields a $(-,-)$-distinguishing proper arc-colouring of $\tilde{D}$, the digraph obtained from D by reversing the direction of each arc. We can thus focus on the $(+,+)$ version of the AVD Conjecture in this section, as these results apply to the $(-,-)$ version as well, through this arc reversing operation.

First off, we note that all digraphs admit a $(+,+)$-distinguishing proper arc-colouring. In other words, we do not need a notion of nice digraphs in this context.

Proposition 2.1. *Every digraph admits a $(+,+)$-distinguishing proper arc-colouring.*

Proof. Let D be any digraph with arcs $a_1,\dots,a_m$. Consider the arc-colouring ϕ that sets $\phi(a_i)=i$ for every $i\in\{1,\dots,m\}$. Clearly, ϕ is proper. It is also $(+,+)$-distinguishing, since, for every vertex u with an out-going arc a_i, only the set $\mathrm{S}^+(u)$ contains i. □

Note that, for any directed cycle D of odd length, we have $\Delta^*(D)=1$, but we need to use $\Delta^*(D)+2=3$ colours (two consecutive arcs must be assigned distinct colours in any $(+,+)$-distinguishing proper arc-colouring of a directed cycle). We believe this is the maximum value of $\mathrm{ndi}_{+,+}(D)$ for a digraph D.

Conjecture 2.2. *For every digraph D, we have* $\mathrm{ndi}_{+,+}(D)\le\Delta^*(D)+2$.

Towards Conjecture 2.2, for every digraph D, we can easily establish an upper bound on $\mathrm{ndi}_{+,+}(D)$ that is linear in $\Delta^*(D)$ by exploiting its relationship with proper edge-colourings of $\mathrm{und}(D)$, the undirected graph underlying D.

Proposition 2.3. *For every digraph D, we have* $\mathrm{ndi}_{+,+}(D)\le 2\Delta^*(D)+2$.

Proof. Let G be the undirected multigraph obtained from D by replacing each arc by an edge. Note that G might indeed have parallel edges, but the maximum multiplicity $\mu(G)$ of its edges is 2. Also, $\Delta(G)\le\Delta^-(D)+\Delta^+(D)\le 2\Delta^*(D)$. Now, by Vizing's Theorem [17], there is a proper edge-colouring ϕ_G of G using $\Delta(G)+\mu(G)=\Delta(G)+2$ colours. We infer ϕ_G to an arc-colouring ϕ_D of D by simply transferring the colour of an edge in G to the corresponding arc in D. By the properness of ϕ_G, note that ϕ_D is also proper. Also, for every arc $\vec{uv}$ of D, we get that no arc out-going from v is assigned colour $\phi_D(\vec{uv})$, and thus, $\mathrm{S}^+(u)\neq\mathrm{S}^+(v)$ since $\phi_D(\vec{uv})\notin\mathrm{S}^+(v)$. Thus, ϕ_D is also $(+,+)$-distinguishing. Since ϕ_D uses at most $2\Delta^*(D)+2$ colours, the result follows. □

Using a different approach, we can slightly improve the upper bound in Proposition 2.3.

Theorem 2.4. *For every digraph D, we have* $\mathrm{ndi}_{+,+}(D)\le 2\Delta^*(D)+1$.

Proof. The proof is by induction on k, the number of vertices with out-degree at least 1. If $k=1$, then D has only one vertex with out-going arcs, in which case D is an out-star. In this case, a $(+,+)$-distinguishing proper $\Delta^*(D)$-arc-colouring ϕ is obtained by assigning a distinct colour to each arc of D. Indeed, ϕ is clearly proper, and we have $\mathrm{S}^+(v)=\emptyset$ for every leaf v of D and $\mathrm{S}^+(u)\neq\emptyset$ for the center u.

Assume the claim holds for k up to some value x, and consider that D has $x+1$ vertices with out-degree at least 1. Let us consider u to be any vertex with out-degree at least 1, and out-going arcs $\vec{uv_1},\dots,\vec{uv_d}$ (where $d\ge1$). Consider $D'=D-\{\vec{uv_1},\dots,\vec{uv_d}\}$ the digraph obtained from D by removing all arcs out-going from u. By the induction hypothesis, D' has a $(+,+)$-distinguishing proper $(2\Delta^*(D)+1)$-arc-colouring $\phi_{D'}$ (since $\Delta^*(D)\ge\Delta^*(D')$), which we would like to extend to a $(+,+)$-distinguishing proper $(2\Delta^*(D)+1)$-arc-colouring ϕ_D of D, *i.e.*, to the arcs $u\vec{v}_1, \dots, u\vec{v}_d$. Note that assigning a colour to any arc $u\vec{v}_i$ only affects $\mathrm{S}^+(u)$. Thus, our goal is to assign colours to the $u\vec{v}_i$'s in a proper way, so that the resulting $\mathrm{S}^+(u)$ is different from $\mathrm{S}^+(w)$ for every neighbour w of u in D. Note that there are at most $2\Delta^*(D)$ such neighbours w around u.

For every arc $u\vec{v}_i$, in terms of properness, the colour assigned to $u\vec{v}_i$ must differ from the colours assigned to the at most $\Delta^*(D)-1$ other arcs in-coming to v_i. Since we are using a set of $2\Delta^*(D)+1$ colours, there are at least $\Delta^*(D)+2$ colours that can, from that point of view, freely be assigned to $u\vec{v}_i$. For every $i \in \{1,\dots,d\}$, let us denote by L_i the set of these colours. Our goal now, is to choose distinct elements (to ensure properness) from $L_1,\dots,L_d$, one from each of the L_i's, in such a way that the union of these elements avoids the sets of colours of the at most $2\Delta^*(D)$ neighbours of u. This is something that can always be done, according to the following claim:

Claim 2.5. *Let $L_1,\dots,L_d$ be $d \geq 1$ sets each containing at least $d+2$ elements. Then, there are at least $2d+1$ different combinations $e_1,\dots,e_d$ of elements, such that $e_i \in L_i$ for every $i \in \{1,\dots,d\}$, and all the e_i's are distinct.*

Proof of the claim. The proof is by induction on d. For $d=1$, we have $|L_1|=3$. Every single element of L_1 is a correct choice as e_1, and thus, there are three correct combinations. Assume now that the claim holds for every d up to some x, and assume $d=x+1$. Without loss of generality, we may assume $1 \in L_1$. Set $L'_2 = L_2 \setminus \{1\},\dots,L'_d = L_d \setminus \{1\}$. By the induction hypothesis, we can produce $2(d-1)+1 = 2d-1$ combinations $e'_2,\dots,e'_d$ of distinct elements from the L'_i's (one from each set). Each such combination $e'_2,\dots,e'_d$, together with 1, yields a combination $1, e'_2,\dots,e'_d$ that is valid for $L_1,\dots,L_d$. Thus, we already know how to generate $2d-1$ different combinations of distinct elements from $L_1,\dots,L_d$, all of which contain the element 1.

All that remains to do is to generate two more combinations. To ensure this, we exhibit two such different combinations not containing the element 1. To that aim, let us choose arbitrary distinct elements $e_2,\dots,e_d$ different from 1 from $L_2,\dots,L_d$. This is possible since each L_i has size at least $d+2$. Now, since L_1 also has size at least $d+2$, there are at least three elements e_1, e'_1, e''_1 that are different from $e_2,\dots,e_d$. At least two of e_1, e'_1, e''_1 must be different from 1. These two elements together with $e_2,\dots,e_d$ yield our remaining two different combinations. $\Diamond$

Now, by Claim 2.5, we can choose a combination $e_1,\dots,e_d$ of distinct elements from $L_1,\dots,L_d$ (where $e_i \in L_i$ for every $i \in \{1,\dots,d\}$) such that no w of the at most $2\Delta^*(D)$ neighbours of u in D verifies $\mathrm{S}^+(w) = \{e_1,\dots,e_d\}$. To finish the extension of $\phi_{D'}$ to ϕ_D, it now suffices to set $\phi_D(u\vec{v}_i) = e_i$ for every $i \in \{1,\dots,d\}$. $\square$

3. The $(+,-)$ version

In this section, we focus on the $(+,-)$ version of the AVD Conjecture, in which, by a proper arc-colouring of a digraph, it is required that $\mathrm{S}^+(u) \neq \mathrm{S}^-(v)$ holds for every arc $u\vec{v}$. Recall that this distinction condition is, out of the four ones we are considering, the closest to the original distinction condition behind the original AVD Conjecture. Indeed, by a proper arc-colouring ϕ of some digraph, for every arc $u\vec{v}$ the colour $\phi(u\vec{v})$ contributes to both $\mathrm{S}^+(u)$ and $\mathrm{S}^-(v)$, which are precisely the two set parameters that are asked to differ for u and v in the $(+,-)$ version.

Compared to the $(+,+)$ version, there are digraphs admitting no $(+,-)$-distinguishing proper arc-colourings. The smallest such digraph is a single arc $u\vec{v}$, as $\mathrm{S}^+(u) = \mathrm{S}^-(v)$ in any proper arc-colouring. Actually, the case of such a pathological arc, already identified in [3], can be generalised in the following way. We say that an arc $\vec{uv}$ of a digraph is *lonely* if $d^+(u) = d^-(v) = 1$. Note that, indeed, $\mathrm{ndi}_{+,-}(D)$ is not defined for every digraph D containing a lonely arc. It turns outs that lonely arcs are the only source of non-colourability in the $(+,-)$ version. That is, if we define a $(+,-)$-*nice digraph* as a digraph D with $\mathrm{ndi}_{+,-}(D)$ being defined, then being $(+,-)$-nice is equivalent to having no lonely arcs.

Proposition 3.1. *A digraph is $(+,-)$-nice if and only if it has no lonely arc.*

Proof. Consider any digraph D, and let us denote by $a_1, \dots, a_m$ its arcs in an arbitrary fashion. Let ϕ be the arc-colouring of D where $\phi(a_i) = i$ for every $i \in \{1, \dots, m\}$. Clearly ϕ is proper. We claim ϕ is $(+,-)$-distinguishing if and only if D has no lonely arc. Indeed, consider an arc $\vec{uv}$. If $d^+(u) \neq d^-(v)$, then for sure $\mathrm{S}^+(u) \neq \mathrm{S}^-(v)$ since $|\mathrm{S}^+(u)| \neq |\mathrm{S}^-(v)|$. Thus, assume $d^+(u) = d^-(v)$. If $d^+(u) = d^-(v) \geq 2$, then, if we denote by v' another out-neighbour of u, we have $\phi(\vec{uv'}) \in S^+(u)$ and $\phi(\vec{uv'}) \notin S^-(v)$, and thus $\mathrm{S}^+(u) \neq \mathrm{S}^-(v)$. The only remaining case is when $d^+(u) = d^-(v) = 1$, which is precisely the case where $\vec{uv}$ is lonely, and we necessarily have $S^+(u) = S^-(v)$ by any proper arc-colouring of D. □

We are actually able to prove a tight general upper bound on $\mathrm{ndi}_{+,-}(D)$ for $(+,-)$-nice digraphs D. We prove that bound right away, because most of the remarks we can raise on the $(+,-)$ version of the AVD Conjecture actually follow from our proof scheme.

Our proof relies on an equivalence between the $(+,-)$ version and particular cases of the original AVD Conjecture, which was already used in [3]. This equivalence is with respect to the following notions and definitions. For a digraph D, by the *bipartite graph associated to* D, we refer to the undirected bipartite graph $B(D)$ constructed as follows:

- The two partite classes of $B(D)$ are V^+ and V^-.
- For every vertex u of D, we add a vertex u^+ to V^+ and a vertex u^- to V^-.
- For every arc $\vec{uv}$ of D, we add the edge u^+v^- to $B(D)$.

In some sense, $B(D)$ is obtained from D by splitting the out-going part and the incoming part of every vertex. Note that $B(D)$ is always balanced, in the sense that $|V^+| = |V^-|$. Also, we can infer some additional properties of $B(D)$ from properties of D.

Observation 3.2. *For every digraph D:*

- $\Delta(B(D)) = \Delta^*(D)$.
- *$B(D)$ is nice if and only if D is $(+,-)$-nice.*

Proof. The first item is because, for every vertex u of D, the value of $d^+(u)$ (in D) equals the value of $d(u^+)$ (in $B(D)$), and similarly $d^-(u)$ equals $d(u^-)$. The second item is because an isolated edge u^+v^- in $B(D)$ corresponds to a lonely arc $\vec{uv}$ in D, and *vice versa.* □

We are now ready to prove our main result in this section.

Theorem 3.3. *The $(+,-)$ version of the AVD Conjecture is equivalent to the (original) AVD Conjecture in bipartite graphs.*

Proof. The notion of the associated bipartite graph is the key behind this equivalence. Indeed, finding a $(+,-)$-distinguishing proper k-arc-colouring of some $(+,-)$-nice digraph is equivalent to finding a distinguishing proper k-edge-colouring of some nice undirected bipartite graph.

- Let D be a $(+,-)$-nice digraph, and consider the (nice, by Observation 3.2) bipartite graph $B = B(D)$ associated to D. Let ϕ_B be a distinguishing proper k-edge-colouring of B. Consider the k-arc-colouring ϕ_D of D obtained by setting $\phi_D(\vec{uv}) = \phi_B(u^+v^-)$ for every arc $\vec{uv}$. Note that ϕ_D is proper because $\phi_D(\vec{uv}) \neq \phi_D(\vec{uv'})$ when $v \neq v'$ since $\phi_B(u^+v^-) \neq \phi_B(u^+v'^-)$ (by the properness of ϕ_B), and, similarly, $\phi_D(\vec{uv}) \neq \phi_D(\vec{u'v})$ when $u \neq u'$ since $\phi_B(u^+v^-) \neq \phi_B(u'^+v^-)$. Also, ϕ_D is $(+,-)$-distinguishing because $\mathrm{S}^+(u) \neq \mathrm{S}^-(v)$ for every arc $\vec{uv}$ of D since, in B, $\mathrm{S}(u^+) \neq \mathrm{S}(v^-)$ (by ϕ_B being distinguishing). Thus, ϕ_D is a $(+,-)$-distinguishing proper k-arc-colouring of D.
- Let B be a nice bipartite graph. Denote by (U, V) the bipartition of B. If necessary, add isolated vertices to B so that 1) B is balanced, and 2) there is an ordering of the vertices in U and V such that $U = \{u_1, \dots, u_n\}$, $V = \{v_1, \dots, v_n\}$, and u_iv_i is not an edge for every $i \in \{1, \dots, n\}$. Under those conditions, let D be the digraph constructed from B by adding a vertex w_i for every $i \in \{1, \dots, n\}$, and an arc $\vec{w_iw_j}$ for every edge u_iv_j of B. Clearly, B is the bipartite graph $B(D)$ associated to D (where U plays the role of V^+ and V plays the role of V^-). We now have the desired equivalence by the arguments used to deal with the previous case. □

The equivalence established in the proof of Theorem 3.3 has a series of consequences on the $(+,-)$ version of the AVD Conjecture. In particular, the fact that $\mathrm{ndi}_{+,-}(D) = \mathrm{ndi}(B(D))$ and $\Delta^*(D) = \Delta(B(D))$ hold for every $(+,-)$-nice digraph D yields some side results. For instance, it is known that there exist nice bipartite graphs G with $\mathrm{ndi}(G) = \Delta(G) + 2$, see [2]. In the same paper, the authors proved that $\Delta(G) + 2$ is actually the maximum value of $\mathrm{ndi}(G)$ for a nice bipartite graph G. In other words, the AVD Conjecture holds for nice bipartite graphs. For our problem, these remarks directly imply:

Corollary 3.4. *For every $(+,-)$-nice digraph D, we have $\mathrm{ndi}_{+,-}(D) \leq \Delta^*(D) + 2$. Furthermore, this upper bound cannot be decreased in general.*

4. The $(-,+)$ version

We now consider the $(-,+)$ version of the AVD Conjecture, in which two vertices u and v that are adjacent through an arc $\vec{uv}$ are required, by a proper arc-colouring, to verify $\mathrm{S}^-(u) \neq \mathrm{S}^+(v)$. Recall that this version is, in some sense, the variant we are considering that is the farthest from the original conjecture. This is mainly because the colour $\phi(\vec{uv})$ of an arc $\vec{uv}$ by an arc-colouring ϕ of a digraph contributes nothing to $\mathrm{S}^-(u)$ and $\mathrm{S}^+(v)$, which are the parameters of u and v that must differ.

In the present context, again, not all digraphs admit $(-,+)$-distinguishing proper arc-colourings. To see this is true, consider the case of a digraph D containing an arc $\vec{st}$ such that $d^-(s) = 0$ (source) and $d^+(t) = 0$ (sink). Clearly, D does not admit a $(-,+)$-distinguishing proper arc-colouring, since we always have $\mathrm{S}^-(s) = \emptyset = \mathrm{S}^+(t)$. Note that the situation remains unchanged if we add the arc $\vec{ts}$ to D, since, here, we would always get $\mathrm{S}^-(s) = \alpha = \mathrm{S}^+(t)$, when assigning a colour α to $\vec{ts}$. If D has two such adjacent vertices, we say that D has a *bad configuration* (such configurations were already considered in [5, 11]). Again, in this variant, a $(-,+)$-*nice digraph* is a digraph D for which $\mathrm{ndi}_{-,+}(D)$ is defined. Actually, bad configurations are the only reason why some digraphs are not $(-,+)$-nice:

Proposition 4.1. *A digraph is $(-,+)$-nice if and only if it has no bad configuration.*

Proof. Let D be any digraph with arcs $a_1, \dots, a_m$, and let ϕ be the arc-colouring of D where $\phi(a_i) = i$ for every $i \in \{1, \dots, m\}$. We claim that ϕ, which is clearly proper, is

$(-,+)$-distinguishing if and only if D has no bad configuration. Indeed, let us focus on an arc $\vec{uv}$. If $d^-(u) \neq d^+(v)$, then $\mathrm{S}^-(u) \neq \mathrm{S}^+(v)$ because $|\mathrm{S}^-(u)| \neq |\mathrm{S}^+(v)|$. So, let us assume that $d^-(u) = d^+(v)$. If $d^-(u) = d^+(v) \geq 2$, then $\mathrm{S}^-(u) \neq \mathrm{S}^+(v)$ due to there being at least one arc out-going from v that is not in-coming to u. The same holds if $d^-(u) = d^+(v) = 1$ and the arc in-coming to u is different from the arc out-going from v. So, there are only two cases remaining: 1) $d^-(u) = d^+(v) = 1$ and $\vec{vu}$ is an arc, and 2) $d^-(u) = d^+(v) = 0$. In both cases, D has a bad configuration, and ϕ cannot be $(-,+)$-distinguishing. □

We note that there are $(-,+)$-nice digraphs D with $\mathrm{ndi}_{-,+}(D) = \Delta^*(D) + 2$. Every odd-length directed cycle is an example of such a digraph. We think this might be the maximum value of $\mathrm{ndi}_{-,+}(D)$ for a $(-,+)$-nice digraph D, which would be reminiscent of the AVD Conjecture.

Conjecture 4.2. *For every $(-,+)$-nice digraph D, we have* $\mathrm{ndi}_{-,+}(D) \leq \Delta^*(D) + 2$.

Towards Conjecture 4.2, contrarily to what was done in Section 2, note that using proper edge-colourings does not yield a linear upper bound (in $\Delta^*(D)$) on $\mathrm{ndi}_{-,+}(D)$ for every $(-,+)$-nice digraph D, as the distinction condition in the $(-,+)$ version, in some sense, asks arcs at distance 2 to be different. Instead, an upper bound on $\mathrm{ndi}_{-,+}(D)$ can be expressed, for instance, as a function of the *strong chromatic index* $\chi'_{\mathrm{s}}(\mathrm{und}(D))$ of $\mathrm{und}(D)$, which is the smallest number of colours in an edge-colouring of $\mathrm{und}(D)$ where no two edges at distance at most 2 get assigned a same colour. Such an upper bound would be quadratic in $\Delta^*(D)$ (see, for instance, [9]).

Using different arguments, for every $(-,+)$-nice digraph D, we provide an upper bound on $\mathrm{ndi}_{-,+}(D)$ that is linear in $\Delta^*(D)$. Just as in Section 3, this is by exploiting some relationship between an arc-colouring of D and an edge-colouring of $B(D)$, the bipartite graph associated to D. However, as highlighted earlier in [5], note that using the relationship is not so natural in the present context. Indeed, by a proper edge-colouring of $B(D)$, having $\mathrm{S}(u^+) \neq \mathrm{S}(v^-)$ is not so relevant regarding D, as, when transposing the edge-colouring to an arc-colouring of D, this would yield $\mathrm{S}^+(u) \neq \mathrm{S}^-(v)$, which is not required in the $(-,+)$ version. Also, it might be that we want $\mathrm{S}^+(u)$ and $\mathrm{S}^-(v)$ to differ in D, while $\mathrm{S}(u^+)$ and $\mathrm{S}(v^-)$ are not required to differ in $B(D)$ because u^+v^- is not an edge. To deal with such issues, we will consider distinguishing proper edge-colourings of $B(D)$ verifying strong distinction conditions.

Before proceeding with the proof, it is important to point out that lonely arcs, though they do not prevent $\mathrm{ndi}_{-,+}(D)$ to be defined for a $(-,+)$-nice digraph D, have a peculiar behaviour (they yield isolated edges in $B(D)$) that will force us to handle them separately. In particular, we will make use of the following property of lonely arcs:

Observation 4.3. *Removing a lonely arc from a digraph cannot create new lonely arcs.*

Proof. Let D be a digraph, and let $D' = D - \vec{uv}$ be the digraph obtained from D by removing a lonely arc $\vec{uv}$. Assume D' has a lonely arc $\vec{xy}$ which is not lonely in D. Then, either $u = x$ or $v = y$. In the first case, we deduce that u has out-degree 2 in D, while, in the second case, we deduce that v has in-degree 2 in D. In both cases, we get a contradiction to the loneliness of $\vec{uv}$. □

We are now ready to prove our main result in this section.

Theorem 4.4. *For every $(-,+)$-nice digraph D, we have* $\mathrm{ndi}_{-,+}(D) \leq 3\Delta^*(D)$.

Proof. Let L be the set of all lonely arcs of D, and set $D' = D - L$. By Observation 4.3, D' has no lonely arcs. Let $B = B(D')$ be the bipartite graph associated to D'. By Observation 3.2, D' is nice, and $\Delta = \Delta(B) \leq \Delta^*(D') \leq \Delta^*(D)$. Recall that the bipartition of B is denoted by (V^+, V^-). Note that B may have several connected components. In what follows, we need to dedicate a special care to some of them. Specifically, a connected component of B is said to be *bad* if it is a star with center in V^+ (and at least two leaves, in V^-, since B is nice). In what follows, we obtain a $(-,+)$-distinguishing proper 3Δ-arc-colouring ϕ_D of D by first colouring the edges of the non-bad connected components of B (and transferring the assigned colours to corresponding arcs in D), then colouring, in D, the arcs corresponding to edges of the bad connected components of B, and eventually colouring the lonely arcs of D.

We start by constructing a proper 3Δ-edge-colouring ϕ_B of the non-bad connected components of B, such that all vertices $v \in V^+$ verify $\mathrm{S}(v) \in \mathcal{X}$ while all vertices $v \in V^-$ verify $\mathrm{S}(v) \in \mathcal{Y}$, for some disjoint sets $\mathcal{X}, \mathcal{Y}$ of subsets of $\{1, \dots, 3\Delta\}$. To achieve this, we split $\{1, \dots, 3\Delta\}$ into the three smaller sets $\mathcal{R} = \{1, \dots, \Delta\}$, $\mathcal{B} = \{\Delta + 1, \dots, 2\Delta\}$, and $\mathcal{G} = \{2\Delta + 1, \dots, 3\Delta\}$ of colours, which we assign as follows.

Let us focus on a non-bad connected component of B. Abusing the notation, let us denote by B this connected component. Recall that there are at least two edges in B, since B is nice. Pick an arbitrary vertex $u^* \in V^+$. This defines a partition of $V(B)$ into *layers* $V_0, \dots, V_d$, where every V_i contains the vertices of B that are at distance exactly i from u^*. Since B is not bad, we have $d \geq 2$. Note that $V_0 = \{u^*\}$, that every edge of B joins vertices in two consecutive V_i's, that every vertex in some V_i with $i \neq 0$ has a neighbour in V_{i-1}, and that the union of all V_i's with even index is exactly V^+ while the union of all V_i's with odd index is exactly V^-. Given an edge uv with $u \in V_i$ and $v \in V_{i+1}$, we say that uv is *downward* from the point of view of u, while it is *upward* from that of v.

For every $i \in \{1, \dots, d\}$, we split V_i into V_i' and V_i'', where V_i' contains the vertices of V_i with no downward edges, while V_i'' contains the vertices in V_i with downward edges. Note that $V_d'' = \emptyset$, and recall that $V_1'' \neq \emptyset$ since B is not bad. We now construct ϕ_B by assigning colours in $\mathcal{R}$, $\mathcal{B}$, and $\mathcal{G}$ to some sets F_R, F_B, and F_G of edges of B, that are defined as follows:

- For all $i \in \{0, \dots, d-1\}$ with $i \equiv 0 \bmod 4$, add all edges in $E(B[V_i \cup V_{i+1}'])$ to F_R, and all edges in $E(B[V_i \cup V_{i+1}''])$ to F_G.
- For all $i \in \{1, \dots, d-1\}$ with $i \equiv 1 \bmod 4$, add all edges in $E(B[V_i \cup V_{i+1}])$ to F_B.
- For all $i \in \{1, \dots, d-1\}$ with $i \equiv 2 \bmod 4$, add all edges in $E(B[V_i \cup V_{i+1}'])$ to F_R, and all edges in $E(B[V_i \cup V_{i+1}''])$ to F_B.
- For all $i \in \{1, \dots, d-1\}$ with $i \equiv 3 \bmod 4$, add all edges in $E(B[V_i \cup V_{i+1}])$ to F_G.

Note that $F_R \cup F_B \cup F_G = E(B)$. Furthermore, each of $B[F_R]$, $B[F_B]$, and $B[F_G]$ induces a subgraph of B with maximum degree at most Δ. By Vizing's Theorem, $B[F_R]$ admits a proper Δ-edge-colouring with colours from $\mathcal{R}$, $B[F_B]$ admits a proper Δ-edge-colouring with colours from $\mathcal{B}$, and $B[F_G]$ admits a proper Δ-edge-colouring with colours from $\mathcal{G}$. Altogether, this yields a proper 3Δ-edge-colouring ϕ_B of B with colours from $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G}$.

In terms of vertex colours, by ϕ_B, we get:

- $\mathrm{S}(u^*)$ has either elements from both $\mathcal{R}$ and $\mathcal{G}$ only (case where $V_1' \neq \emptyset$ and $V_1'' \neq \emptyset$) or elements from $\mathcal{G}$ only (case where $V_1' = \emptyset$).

- Consider a vertex $u \in V_1$. On the one hand, if $u \in V_1'$, then $\mathrm{S}(u)$ contains only one element, from $\mathcal{R}$. On the other hand, if $u \in V_1''$, then $\mathrm{S}(u)$ contains exactly one element from $\mathcal{G}$ and at least one element from $\mathcal{B}$ (and no element from $\mathcal{R}$).
- More generally, for every $u \in V_i$ with $i > 1$:
  - Assume $i \equiv 0 \bmod 4$. On the one hand, if $u \in V_i'$, then $\mathrm{S}(u) \subseteq \mathcal{G}$. On the other hand, if $u \in V_i''$, then $\mathrm{S}(u)$ contains at least one element from $\mathcal{G}$, and perhaps elements from $\mathcal{R}$ (and no elements from $\mathcal{B}$).
  - Assume $i \equiv 1 \bmod 4$. On the one hand, if $u \in V_i'$, then $\mathrm{S}(u) \subseteq \mathcal{R}$. On the other hand, if $u \in V_i''$, then $\mathrm{S}(u)$ contains at least one element from $\mathcal{G}$, and at least one element from $\mathcal{B}$ (and no elements from $\mathcal{R}$).
  - Assume $i \equiv 2 \bmod 4$. On the one hand, if $u \in V_i'$, then $\mathrm{S}(u) \subseteq \mathcal{B}$. On the other hand, if $u \in V_i''$, then $\mathrm{S}(u)$ contains at least one element from $\mathcal{B}$, and perhaps elements from $\mathcal{R}$ (and no elements from $\mathcal{G}$).
  - Assume $i \equiv 3 \bmod 4$. On the one hand, if $u \in V_i'$, then $\mathrm{S}(u) \subseteq \mathcal{R}$. On the other hand, if $u \in V_i''$, then $\mathrm{S}(u)$ contains at least one element from $\mathcal{B}$, and at least one element from $\mathcal{G}$ (and no elements from $\mathcal{R}$).

It is then easy to check that ϕ_B is distinguishing, since vertices in V^+ and in V^- have sets of colours in disjoint sets $\mathcal{X}$ and $\mathcal{Y}$. In particular, the fact that $V_1'' \neq \emptyset$ (because B is not bad) implies that $\mathrm{S}(u^*)$ contains at least one element from $\mathcal{G}$.

By applying the arguments above to all the non-bad connected components of the whole of B, we get ϕ_B, a partial distinguishing proper 3Δ-edge-colouring of the non-bad connected components of B. By arguments used in the proof of Theorem 3.3, the properties of ϕ_B remain when transforming ϕ_B to a partial arc-colouring $\phi_{D'}$ of D'. In particular, $\phi_{D'}$ is proper and uses at most 3Δ colours, and, for every arc $\vec{uv}$ such that at least one of u and v does not belong to a bad connected component of B, we have $\mathrm{S}^-(u) \neq \mathrm{S}^+(v)$.

Our goal now is to extend $\phi_{D'}$ to a $(-,+)$-distinguishing proper $(3\Delta^*(D))$-arc-colouring ϕ_D of D. To that aim, note that two types of arc configurations remain to be coloured in D: 1) the configurations corresponding to the bad connected components of B, and 2) the lonely arcs. Extending the colouring to such configurations can be proved to always be possible, via, essentially, counting arguments.

First, consider a bad connected component in B. By definition, this connected component is a star with unique vertex u^+ in V^+ being its center, and being adjacent to $k \geq 2$ leaves $v_1^-, \dots, v_k^-$ in V^-. Back in D, this corresponds to a vertex u with out-neighbours $v_1, \dots, v_k$, where u has no other out-going arcs while all the v_i's have no other in-coming arcs. For every v_i, note that if all arcs in-coming to u and all arcs out-going from v_i have already been coloured, then $\mathrm{S}^-(u) \neq \mathrm{S}^+(v_i)$. This is because either both u^- and v_i^+ belong to non-bad connected components of B (in which case the distinction comes from how B was edge-coloured), because v_i^+ is the center of a bad connected component in B whose associated bad configuration in D was treated earlier (in which case the distinction comes from the upcoming counting arguments), or because u^- is a leaf in a bad connected component of B whose associated bad configuration in D was treated earlier (in which case the distinction comes from similar arguments as in the previous case). Thus, when colouring the arcs $u\vec{v}_1, \dots, u\vec{v}_k$, we only need to make sure that 1) all arcs out-going from u are assigned distinct colours, that 2) for every v_i and every out-neighbour w of v_i, the resulting set $\mathrm{S}^-(v_i)$ is distinct from the set $\mathrm{S}^+(w)$, and that 3) the resulting set $\mathrm{S}^+(u)$ is distinct from the set $\mathrm{S}^-(w)$ of every in-neighbour w of u. Since $d^+(v_i) \leq \Delta^*(D)$ for every

v_i, regarding the second condition, there is a set L_i of at least $2\Delta^*(D)$ colours that can be freely assigned to $\vec{uv_i}$ without violating that condition. We are now in a weaker condition than that of the statement of Claim 2.5: we have sets $L_1, \dots, L_d$ of at least $2\Delta^*(D)$ elements, and we have to find a combination $e_1, \dots, e_d$ of their elements such that all e_i's are distinct, each e_i lies in L_i, and the set $\{e_1, \dots, e_d\}$ is different from $\mathrm{S}^-(w)$ for each w of the at most $\Delta^*(D)$ in-neighbours w of u in D. By Claim 2.5, there is a such combination $e_1, \dots, e_d$, and we can correctly extend the arc-colouring by setting $\phi_D(\vec{uv_i}) = e_i$ for every $i \in \{1, \dots, d\}$.

Once no configuration corresponding to a bad connected component of B remains in D, we are left with assigning a colour to each of the lonely arcs in L. Let us focus on a lonely arc $\vec{uv}$ of L that remains to be coloured. Recall that having $\mathrm{S}^-(u) \neq \mathrm{S}^+(v)$ does not depend on how $\phi_D(\vec{uv})$ is chosen; that distinction condition is either already met by previous colouring arguments, or will be met by how other lonely arcs will be coloured later on. By definition of a lonely arc, recall that $\vec{uv}$ is the only arc in-coming to v and the only arc out-going from u. Hence, at the moment, $\mathrm{S}^+(u) = \mathrm{S}^-(v) = \emptyset$, and assigning a colour to $\vec{uv}$ will completely determine $\mathrm{S}^+(u) = \mathrm{S}^-(v)$. Also, since v has no other arc coming in and u has no other arc going out, assigning any colour to $\vec{uv}$ cannot break the properness of ϕ_D. So, we just need to assign any colour to $\vec{uv}$ so that $\mathrm{S}^+(u)$ is different from $\mathrm{S}^-(w)$ for every w such that $\vec{wu}$ is an arc, and so that $\mathrm{S}^-(v)$ is different from $\mathrm{S}^+(w)$ for every w such that $\vec{vw}$ is an arc. There are at most $\Delta^*(D)$ such w for u, while there are at most $\Delta^*(D)$ such w for v, hence at most $2\Delta^*(D)$ constraints in total. Since we are using a set of $3\Delta^*(D)$ colours, there is a free one that we can assign to $\vec{uv}$ by ϕ_D, without raising conflicts in terms of properness or distinction.

Once all lonely arcs of L have been treated that way, ϕ_D is a $(-,+)$-distinguishing proper $(3\Delta^*(D))$-arc-colouring of D. $\square$

5. Conclusion

Our goal in this paper was to investigate directed counterparts of the AVD Conjecture where, by a proper arc-colouring of a digraph, adjacent vertices are required to be distinguished by a given one of their two set parameters. For each of the four resulting variants, we believe that, for any nice (specific to the variant) digraph D, there should be a proper $(\Delta^*(D) + 2)$-arc-colouring which is as desired. We verified this for the $(+,-)$ variant (Corollary 3.4), while we only verified weaker statements for the other variants (Theorems 2.4 and 4.4).

An interesting aspect of this line of research lies in the differences between the original AVD Conjecture and each of the four variants, and also in the differences between these four variants. For instance, the notion of nice digraphs varies greatly from one variant to another. Also, the effects of colouring an arc in the four variants are more or less distant from the effects of colouring an edge in the original conjecture. In terms of inherent hardness, the $(+,-)$ version of the AVD Conjecture seems to be the easiest one, as we proved it is equivalent to a very restricted case of the original conjecture (Theorem 3.3). We have the feeling that the $(-,+)$ version should be the hardest one, particularly due to the fact that the colouring mechanisms are a bit less local.

Regarding further work, of course the most important direction would be to tackle Conjectures 2.2 and 4.2. It could also be interesting to investigate our four variants of the AVD Conjecture in restricted classes of digraphs such as tournaments or acyclic digraphs. Lastly, as mentioned earlier, distinguishing adjacent vertices by a single set parameter is only one possible way for defining a directed counterpart to the AVD Conjecture, but playing with combinations of the two set parameters, just like Sopena and Woźniak did in [15], might lead to other interesting problems as well.

References

[1] S. Akbari, H. Bidkhori, N. Nosrati. r-strong edge colorings of graphs. *Discrete Mathematics*, 306:3005-3010, 2006.

[2] P.N. Balister, E. Győri, J. Lehel, R.H. Schelp. Adjacent vertex distinguishing edge-colorings. *SIAM Journal on Discrete Mathematics*, 21:237-250, 2007.

[3] E. Barme, J. Bensmail, J. Przybyło, M. Woźniak. On a directed variation of the 1-2-3 and 1-2 Conjectures. *Discrete Applied Mathematics*, 217:123-131, 2017.

[4] O. Baudon, J. Bensmail, É. Sopena. An oriented version of the 1-2-3 Conjecture. *Discussiones Mathematicae Graph Theory*, 35(1):141-156, 2015.

[5] J. Bensmail, K. Szabo Lyngsie. 1-2-3 Conjecture in Digraphs: More Results and Directions. *Discrete Applied Mathematics*, 284:124-137, 2020.

[6] M. Borowiecki, J. Grytczuk, M. Pilśniak. Coloring chip configurations on graphs and digraphs. *Information Processing Letters*, 112:1-4, 2012.

[7] A.C. Burris, R.H. Schelp. Vertex-distinguishing proper edge-colorings. *Journal of Graph Theory*, 26(2):73-82, 1997.

[8] J. Černy, M. Horňak, R. Soták. Observability of a graph. *Mathematica Slovaca*, 46:21-31, 1996.

[9] P. Erdős, J. Nešetřil. *Irregularities of partitions.* G. Halász, V.T. Sós, Eds., [Problem], 162–163, 1989.

[10] I.J. Holyer. The NP-completeness of edge coloring. *SIAM Journal on Computing*, 10:718-720, 1981.

[11] M. Horňak, J. Przybyło, M. Woźniak. A note on a directed version of the 1-2-3 Conjecture. *Discrete Applied Mathematics*, 236:472-476, 2018.

[12] D. König. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Mathematische Annalen*, 77(4):453–465, 1916.

[13] H. Li, Y. Bai, W. He, Q. Sun. Vertex-distinguishing proper arc colorings of digraphs. *Discrete Applied Mathematics*, 209:276-286, 2016.

[14] B. Seamone. The 1-2-3 Conjecture and related problems: a survey. Preprint, 2012. Available online at `http://arxiv.org/abs/1211.5122`.

[15] É. Sopena, M. Woźniak. A note on the neighbour-distinguishing index of digraphs. Preprint, 2019. Available online at `http://arxiv.org/abs/1909.10240`.

[16] V.G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz.*, 3:25-30, 1964.

[17] V.G. Vizing. The chromatic class of a multigraph. *Kibernetika*, 3:29-39, 1965.

[18] D.B. West. *Introduction to Graph Theory.* Prentice Hall, New Jersey, 1996.

[19] Z. Zhang, L. Liu, J. Wang. Adjacent strong edge coloring of graphs. *Applied Mathematics Letters*, 15:623-626, 2002.